J. Fluid Mech. (1986), vol. 163, pp. 59–67
Printed in Great Britain

The motion of a viscous drop sliding down a Hele-Shaw cell

By KALVIS M. JANSONS
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW

(Received 28 February 1985 and in revised form 1 October 1985)

The motion of a viscous drop in a vertical Hele-Shaw cell is studied in a limit where the effect of surface tension through contact-angle hysteresis is significant. It is found that a rectangular drop shape is a possible steady solution of the governing equations, although this solution is unstable to perturbations on the leading edge. Even though the unstable edge is one where a viscous fluid is moving into a less viscous fluid, in this case air, this is shown to be a special case of the well-known Saffman–Taylor instability. An experiment is performed with an initially circular drop in which it is observed that the drop shape becomes approximately rectangular except at the leading edge, where it becomes rounded and sometimes has a ragged appearance.

A drop sliding down a vertical Hele-Shaw cell is an example of a system where the action of surface tension is not always one of smoothing, since in this case it leads to the formation of right-angle corners at the back of the drop (rounded only slightly on the lengthscale of the gap thickness of the cell).

1. Introduction

The motion of a viscous drop driven by gravity in a vertical Hele-Shaw cell is considered, where a Hele-Shaw cell consists of two parallel plates with separation very much smaller than the lengthscale of interest along the plates, and so the system may be considered as two-dimensional. The motion is assumed to be sufficiently slow for the zero-Reynolds-number (or Stokes flow) approximation to be appropriate and the effect of surface tension through the difference between the advancing and receding contact angles is assumed to be significant. A contact angle is informally the angle between the bounding solid surface and the tangent plane of the fluid–fluid interface at the line of three-phase contact (or contact line). For a more precise definition of these terms see Jansons (1985). For simplicity we shall consider only the case of a viscous drop sliding through an air-filled Hele-Shaw cell, where the dynamical effect of air will be completely ignored.

The motivation for this study was to improve understanding of the effect of contact-angle hysteresis (i.e. the discontinuity between advancing and receding contact angles) on the motion of a viscous drop sliding down an inclined plane (Dussan V. & Chow 1983). The case of a drop in a Hele-Shaw cell was chosen because of its simplicity. The problem considered in this paper is also related to the removal ('washout') of a drop of one fluid from either a porous medium or a thin gap between solid surfaces, by flushing it out with another fluid. A more closely related problem still is the removal of a drop of fluid from a thin gap by a centrifuge.

Problems of this sort are important in a variety of fields, for example oil recovery and the movement of large amounts of contaminants in lubrication films. It will also

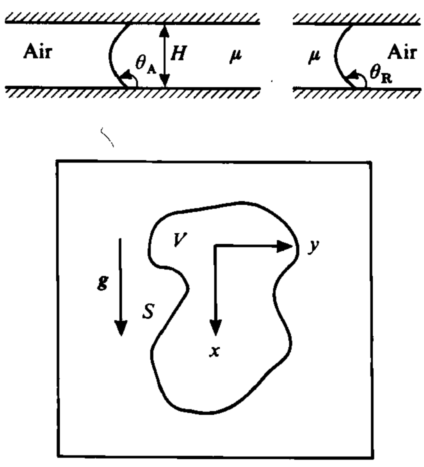


FIGURE 1. Cross-sectional and plan view of Hele-Shaw cell.

be seen that the study of the motion of a viscous drop in a vertical Hele-Shaw cell is interesting in its own right, showing for example that the effect of surface tension on the drop shape is not always a smoothing influence and can result in the formation of sharp angles.

In §2 we begin by deriving the equations and boundary conditions for a drop in a Hele-Shaw cell. The boundary conditions require some careful thought as the Navier–Stokes equation together with the no-slip boundary condition predict a non-integrable stress at the contact line (Jansons 1985). Jansons (1986) discusses the viscous-stress singularity in detail and shows that, in the case of a rough surface, for a sufficiently slowly moving contact line the singularity can be removed within continuum theory. In the second part of this section we show that a rectangular drop shape is a possible steady solution of the governing equations and argue that the back of the drop is stable and the front unstable. Even though the instability occurs when the more viscous fluid is driven into the less viscous air, it is shown that it is in fact the Saffman–Taylor instability.

In §3 we consider an experiment which begins with a circular drop of black treacle and shows that the drop shape tends to become approximately rectangular, except at the leading edge where it is rounded and sometimes has a ragged appearance.

2. Theory

In this section we derive the governing equations and boundary conditions for a viscous drop sliding down a vertical Hele-Shaw cell and use these equations to find a family of steady solutions. We shall begin by considering the limit of small capillary number $\mathbb{C}$, where the capillary number is defined as the ratio of viscous forces to surface-tension forces. However, it is shown later that the results obtained can be easily generalized to the case of arbitrary capillary number, and even to systems where there is not strictly a moving contact line at all, namely where a thin film of fluid is left behind the drop. In this case the receding contact angle can be taken as π. Note that we are using the convention that the contact angle is measured from the solid surface behind the contact line to the fluid–fluid interface (see figure 1).

2.1. *Governing equations and boundary conditions*

Define Cartesian axes with the x-axis parallel to the direction of gravity $\boldsymbol{g}$ and the y-axis in the plane of the Hele-Shaw cell as shown in figure 1. Assuming that the characteristic dimension L of the drop in the plane of the cell is very much greater than the gap thickness H, we may describe the motion of the drop by lubrication theory. In this case, the equations of lubrication theory reduce to Darcy's law (Richardson 1981), namely

$$\boldsymbol{u} = -\alpha \nabla (p - \rho g x), \tag{1}$$

where $\alpha = H^2/12\mu$, ρ is the fluid density, p is the fluid pressure, $\boldsymbol{u}$ is the fluid velocity averaged over z and ∇ is the gradient operator in the (x, y)-plane. Define $V = \alpha \rho g$, which is the fall speed of a drop with zero surface tension; then we may write (1) as

$$\boldsymbol{u} = -\alpha \nabla p + V \hat{\boldsymbol{x}}, \tag{2}$$

where $\hat{\boldsymbol{x}}$ is the unit vector in the direction of increasing x.

In the case where the capillary number $\mathbb{C} = \mu u_0/\gamma$ (where u_0 is a typical value of $|\boldsymbol{u}|$) is very much less than unity there exist well-defined receding and advancing contact angles, independent of the velocity, and the interface between the drop and the air may be approximated by an arc of a circle (Jansons 1986). This implies that there is a pressure jump independent of z at the fluid–air interface that is also independent of velocity. If we take the pressure in the air as zero we may define the pressure just inside the drop as p_{A} for the advancing part of the contact line (i.e. the viscous fluid moving into the air) and p_{R} for the receding part of the contact line. These are given in terms of the advancing and receding contact angles θ_{A} and θ_{R} by

$$p_{\mathrm{A,\,R}} = -\frac{2\gamma}{H} \cos \theta_{\mathrm{A,\,R}} \, (1 + O(KH)), \tag{3}$$

where K (typically $O(L^{-1})$) is the curvature of the free surface in the plane of the drop. Contact-angle hysteresis implies that p_{A} is not equal to p_{R} and in fact $p_{\mathrm{A}} > p_{\mathrm{R}}$ in practice, which is a result that can be proved by a thermodynamic argument. There must also be a transition region where the interface is neither advancing nor receding (Dussan V. & Chow 1983); in this region the pressure is between p_{A} and p_{R} and the appropriate boundary condition is $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, where $\boldsymbol{n}$ is a unit normal to the fluid–air interface in the plane of the Hele-Shaw cell.

One might hesitate before applying the above boundary conditions for the pressure directly to (2), because of the classical viscous-stress singularity at the contact line (Jansons 1985). However, this is not a problem since it is often found that, in moving-contact-line problems in which the lubrication approximation may be used everywhere except at the contact line, the viscous singularity can be ignored by using the lubrication equations even in the neighbourhood of the contact line (Buckmaster 1977; Huppert 1982). The nature of the viscous-stress singularity, which does not interest us here, is considered in detail by Jansons (1986) and only those results relevant to the current problem will be mentioned here.

The net viscous force per unit length of the contact line on a region of size $O(H)$ about the contact line is of the order $\mu U \log (H/\delta)$, where U is the normal speed of the contact line and δ is the fundamental cutoff lengthscale of the viscous-stress singularity (Jansons 1986). For this viscous force in the neighbourhood of the contact line to be negligible we require it to be very much smaller than the pressure jump, $O(\gamma/H)$, at the fluid–air interface. This implies that $\mathbb{C} \log (H/\gamma)$ must be very much less than unity. To interpret this condition we must estimate the lengthscale δ.

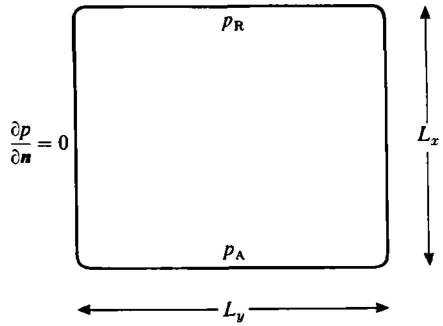


FIGURE 2. Diagram of steady-state drop shape.

Jansons (1986) argued that δ is in the range from molecular dimensions up to macroscopic dimensions (in this case H). It was also shown that δ is velocity-dependent on rough surfaces, and that in the limit of small capillary number δ is very much larger than roughness dimensions. In practice all solid surfaces are rough; however, the details of the velocity dependence of δ are only of minor interest here, since even with δ of molecular dimensions the condition $\mathbb{C} \log (H/\delta) \ll 1$ is not much stronger than $\mathbb{C} \ll 1$, as already assumed. Hence for the remainder of this paper we shall assume that the viscous-stress singularity is ignorable in the limit $H \ll L$, where the contact-line region is only a small part of the whole drop.

For use in the rest of this section we non-dimensionalize p, $\boldsymbol{u}$ and $\boldsymbol{x}$ with respect to γ/H, V and L. Then (2) becomes

$$\boldsymbol{u} = -\mathbb{B}^{-1} \nabla p + \hat{\boldsymbol{x}}, \tag{4}$$

where the parameter $\mathbb{B} = \rho g L H/\gamma$ is the Bond number based on the lengthscale $(HL)^{\frac{1}{2}}$. For the drop to move at all, we require that $\mathbb{B}$ is greater than or of the order of unity. In addition to this we also require that $H\mathbb{B}/L \ll 1$, since $\mathbb{C} = O(H\mathbb{B}/L)$ and $\mathbb{C}$ is assumed to be very much less than unity.

2.2. *A family of steady solutions*

A family of steady solutions to the governing equation (4) interestingly is made up of rectangles of arbitrary aspect ratio, aligned with the x-axis. However, the rectangular solutions of (4) are not valid in an $O(H)$ region about the corners.

Define L_x and L_y as the lengths of the vertical and horizontal sides of the rectangular drop (see figure 2). Then these lengths must satisfy $L_x L_y H$ equal to the volume of the drop, which is assumed constant. The boundary condition on the vertical sides, $\boldsymbol{u} \cdot \boldsymbol{n} = 0$, may be written as $\boldsymbol{n} \cdot \nabla p = 0$. Then for a drop that is sliding down the Hele-Shaw cell we find

$$p = p_R + (p_A - p_R) \frac{x}{L_x}, \tag{5}$$

where we have taken the receding edge at $x = 0$. The z-averaged velocity field is constant inside the drop, and is equal to

$$\left(1 - \mathbb{B}^{-1} \frac{p_A - p_R}{L_x}\right) \hat{\boldsymbol{x}}. \tag{6}$$

Note that (6) implies that if $\mathbb{B} \leqslant (p_A - p_R)/L_x$ then no motion will occur.

It is interesting to note that the rectangular-drop solutions to the governing equations would still be valid if p_A and p_R were functions of the normal velocity of the contact line, and so would be valid for arbitrary capillary number, since the velocity generated along the advancing and receding edges is independent of the position along the contact line. However in this case (3), for the pressure jump, is invalid. It has been shown by Bretherton (1961) that at a sufficiently high capillary number a drop changes from having a moving contact line to leaving a thin film of fluid. In the case of a viscous drop moving in air this is likely to happen only at the receding contact line. However, if the film left behind the drop does not result in a large rate of change of volume, we can define an 'effective' contact line at the beginning of the thin film. In this case the rectangular-drop solutions are still valid, since there is a well-defined pressure jump at the effective contact line.

Stability of the rectangular drop

Unfortunately, the rectangular-drop solution considered above is unstable to perturbations on the leading edge. The nature of the instability can be analysed, showing that it is of the Saffman–Taylor type. This might appear surprising at first sight, since at the leading edge the more viscous fluid is moving into a region occupied by the less viscous air, which would seem to contradict the usual conditions for the Saffman–Taylor instability. However, this is not the case, as will now be shown.

The contribution to the velocity field inside the drop directly from gravity is uniform throughout the drop, regardless of drop shape, and is of magnitude V. This contribution cannot therefore give rise to an instability. To understand the instability, and to relate it to that studied by Saffman & Taylor (1958), it is helpful to consider the drop in a frame moving with velocity $V\hat{\boldsymbol{x}}$; this transforms (2) into the usual form of Darcy's law. In this frame all the fluid motion is driven by the pressure gradient due to contact-angle hysteresis. For a rectangular drop this pressure gradient is uniform and is directed from the back to the front.

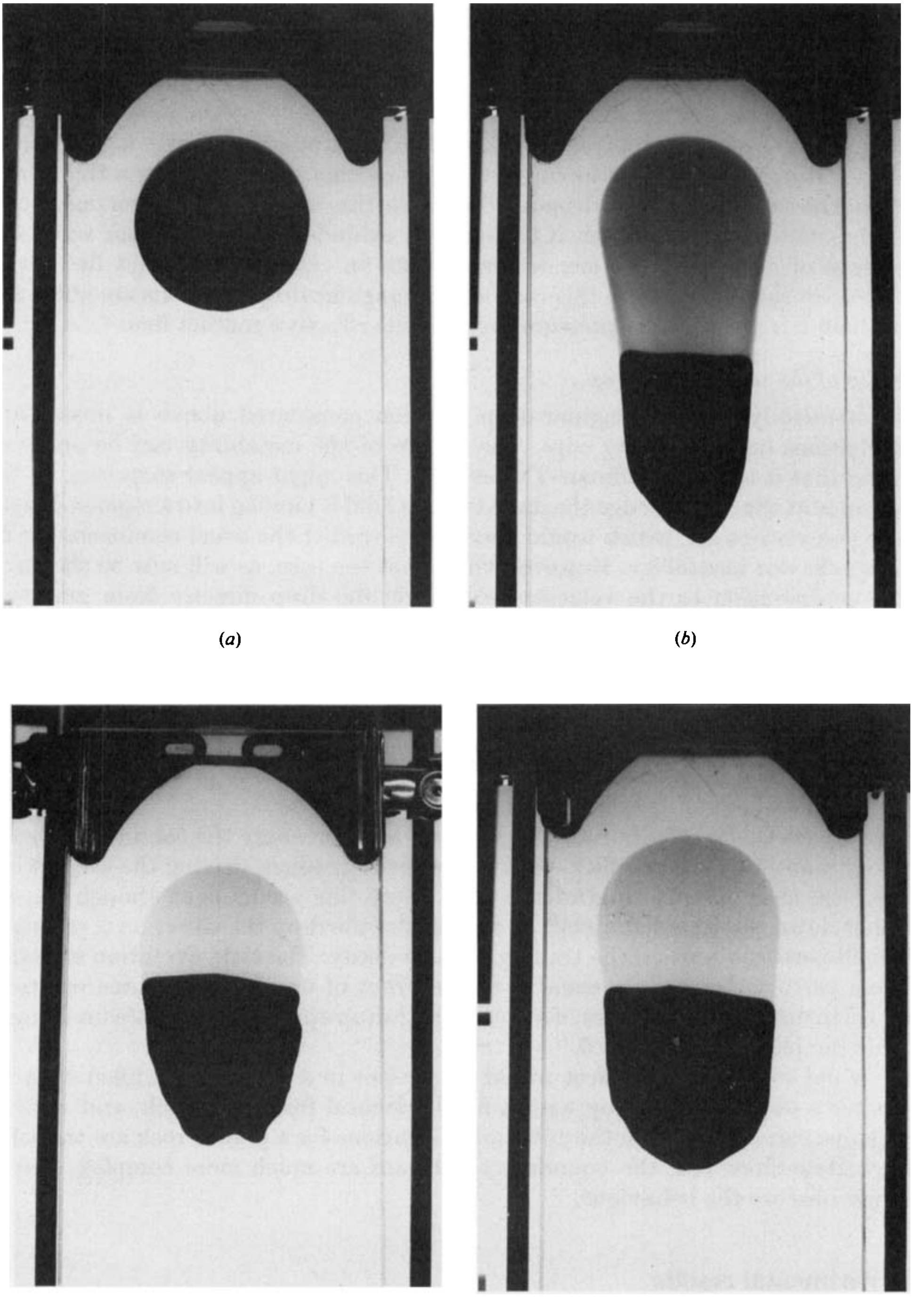
This implies that at the front of the drop we have precisely the conditions for the usual Saffman–Taylor instability, with the pressure gradient driving the less viscous air into the more viscous fluid of the drop. Note this occurs even though the *net* motion is down the Hele-Shaw cell. At the back of the drop the situation is reversed, and small perturbations on the trailing edge do decay. The time-evolution equation for these perturbations is the same as for the front of the drop with time reversed. Small perturbations on the sides of a rectangular drop appear to translate unchanged towards the back, since $\boldsymbol{u}\cdot\boldsymbol{n} = 0$.

One wonders if a similar effect would be present in a porous rock filled with air except for a large viscous drop with a flat horizontal front and back, and vertical sides. However, even though the governing equations for a porous rock are the same as for a Hele-Shaw cell, the boundary conditions are much more complex, and so this may obscure the behaviour.

3. Experimental results

An experiment was performed to determine whether the time-dependent motion changed an initially circular drop sliding through a Hele-Shaw cell into a drop of approximately rectangular shape. However, we did not expect the front of the drop to appear rectangular because of the instability of the leading edge, and it was not even clear whether a steady-state drop shape would exist.

The experiment was simple and could have been performed in the average kitchen.



(*a*) (*b*) (*c*) (*d*)

FIGURE 3. Experimental results. (*a*) Initial state of run 1 with 2 mm gap and diameter 9 cm; (*b*) run 1 after 15 minutes; (*c*) run 2 after 50 minutes with 1 mm gap and initial diameter 9 cm; (*d*) run 3 after 50 minutes with 1 mm gap and initial diameter 9.5 cm.

Two Perspex plates were cleaned with washing-up liquid, thoroughly rinsed with water and allowed to dry. These plates were then assembled into a Hele-Shaw cell trapping a 9–9.5 cm drop of black treacle. The drop naturally took an initially circular shape since the gap thickness of the Hele-Shaw cell was less than the height of the drop before the second Perspex plate was put into position. An approximately constant separation was maintained by inserting small metal plates at the corners of the Perspex plates, which also allowed for the alteration of the gap for different experiments. The experiment was performed once with a 2 mm gap thickness and twice with a 1 mm gap.

The surface tension and viscosity of the black treacle used in the experiment were difficult to measure and were also very sensitive to temperature. However, the following estimates were made: viscosity $10^3\ \mathrm{g\ cm^{-1}\ s^{-1}}$, surface tension $60\ \mathrm{dyne\ cm^{-1}}$ and density $1\ \mathrm{g\ cm^{-3}}$. The advancing and receding contact angles were also estimated as $\frac{1}{2}\pi$ and π. Strictly, the receding contact angle did not exist, since at the rear of the drop in the experiment a thin film was left behind. However, because the production of the thin film did not result in a significant rate of change of volume it can be ignored, provided that the correct pressure jump is used as a boundary condition. In the limit of small capillary number this can be achieved by taking the effective contact angle equal to π.

The experimental results are given in figure 3, which consists of photographs from three distinct runs. The best examples of the rectangular structure at the rear of the drop are shown in figures 3(*b*) and (*d*), which illustrate the prediction of right-angle corners at the back. Figure 1 (*c*) is included to show that this experiment is not always reproducible, since it had almost the same initial conditions and was allowed to run for the same time as the drop in figure 3(*d*). However, the differences here are probably principally due to the effects of contamination on the two bounding solid surfaces of the Hele-Shaw cell, rather than the instability at the leading edge of the drop.

It would be interesting to repeat these experiments under better conditions to see if the motion of an initially circular drop is deterministic, or is highly sensitive to perturbations to the initial conditions.

Figures 3(*b–d*) also show the formation of a thin film at the rear edge of the drop, as mentioned earlier in this section. The velocity dependence of the film thickness is apparent in figure 3 from the darkness of the film region; compare the films in figures 3(*b*) and (*d*) for example. Later, in all the experiments the film showed signs of instability resulting in the formation of dry patches in its interior, and, since the film was estimated to be too thick (greater than 0.01 mm) for van der Waals forces to be directly responsible, the break-up of the film was most likely to have been due to surface-tension gradients resulting from local contamination of the surface.

4. Conclusion and discussion

We have shown that a drop sliding through a Hele-Shaw cell exhibits some interesting behaviour resulting from the effect of contact-angle hysteresis, and that a steady solution to the governing equations is a rectangular drop shape. However, the front of the rectangular drop was shown to be unstable, and the instability was identified as the one first analysed by Saffman & Taylor (1958). It is interesting to recall that the unstable edge is one where the more viscous fluid is moving into the less viscous air, and this was explained by observing that the pressure gradient due to the contact-angle hysteresis was responsible and that this is directed to oppose the motion of the drop.

It is not clear whether the motion of an initially circular drop is deterministic, since, although the front of the rectangular drop is unstable, the front of the circular drop may be sufficiently close to the final single finger that appears in the Saffman–Taylor instability for further fingering to be suppressed. However, to resolve this point either a better experiment or a numerical simulation is required.

Even though we began by assuming that the capillary number was very much less than unity it was shown that the rectangular drop shape is a valid steady solution to the governing equations for arbitrary capillary number. This was because both at the front and back of the drop the normal velocity of the contact line was independent of the position along it, and, therefore, so was the pressure jump at the contact line. This is even the case when a thin film of fluid is left behind the drop if the rate of change of volume of the drop due to the loss of fluid to the thin film is sufficiently small that it can be neglected, which is precisely the case at the rear of the drops in the experiments. In this case an effective contact line is the beginning of the thin film and the effective contact angle is π. However, if the initially circular drops considered in the experiments do tend to a steady state at all, this final drop shape will depend on the pressure jump as a function of normal velocity of the contact line, since the front of the drop cannot be straight, for this would be unstable.

In all sliding-drop problems where contact-angle hysteresis is present the contact line can be split into three distinct regions (not necessarily connected). These are where the contact line is (i) advancing, (ii) not moving and (iii) receding. This classification of points on the contact line is applicable to both a drop sliding down a Hele-Shaw cell and a drop sliding down an inclined plane, and one may be used as an aid to understanding the mechanisms in the other. In region (ii), where the contact line is not in motion, the contact angle is allowed to change continuously between its advancing and receding values. In the case of a drop sliding down a Hele-Shaw cell the point of transition between regions (ii) and (iii) is much more clearly visible than the corresponding point for a drop sliding down an inclined plane, since the contact line turns through approximately a right angle within the space of the gap thickness of the cell. The transition point between regions (i) and (ii) is less pronounced and resembles that for a drop on an inclined plane more closely.

The similarity between the equations of a Hele-Shaw cell and those of a porous medium, although the boundary conditions are more complex in the latter, suggest that a drop sliding through a porous medium may show similar characteristics to the drop considered in this study. For example, a drop sliding through a porous medium might form a flattened back (again ignoring the fluid left behind the main drop as was done in the case of the thin film in the Hele-Shaw cell).

The instability at the front of a drop sliding through a Hele-Shaw cell may be related to the instability at the front of a sheet of viscous fluid in the experiment of Huppert (1982). This experiment consisted of an inclined plane with a viscous fluid initially held at the top of the plane by means of a barrier. Since the horizontal lengthscale was large compared with the lengthscale down the plane the motion was approximately two-dimensional after the barrier was removed, until the appearance of an instability at the leading edge. One important difference between this experiment and a drop sliding down a Hele-Shaw cell is that the thickness of the layer is not constrained. However, it is possible that the reason for the instability is still due to the contribution to the velocity field opposing the bulk motion resulting from the effect of contact-angle hysteresis.

REFERENCES

BRETHERTON, F. P. 1961 The motion of long bubbles in tubes. *J. Fluid Mech.* **10**, 166.

BUCKMASTER, J. D. 1977 Viscous sheets advancing over dry beds. *J. Fluid Mech.* **81**, 735.

DUSSAN V., E. B. & CHOW, R. T.-P. 1983 Sticking of drops or bubbles to solid surfaces. *J. Fluid Mech.* **137**, 1.

HUPPERT, H. E. 1982 Flow and instability of a viscous current down a slope. *Nature* **300**, 427.

JANSONS, K. M. 1985 Moving contact lines on a two-dimensional rough surface. *J. Fluid Mech.* **154**, 1.

JANSONS, K. M. 1986 Moving contact lines at non-zero capillary number. Submitted to *J. Fluid Mech.*

RICHARDSON, S. 1981 Some Hele-Shaw flows with time-dependent free boundaries. *J. Fluid Mech.* **102**, 263.

SAFFMAN, P. G. & TAYLOR, G. I. 1958 The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proc. R. Soc. Lond.* A **245**, 312.